Intro. to ODEs

Quiz 4 **Solutions**

1) Find the general solution to the following differential equation.

$$\left(\frac{y}{x} + 6x\right) dx + (\ln(x) - 2)\, dy = 0$$

Since $\frac{\partial}{\partial y}\left(\frac{y}{x} + 6x\right) = \frac{1}{x} = \frac{\partial}{\partial x}(\ln(x) - 2)$, we know that the equation is exact (it also happens to be first order linear). So, the general solution is of the form $F(x, y) = C$ where F must have the form

$$F(x, y) = \int \left(\frac{y}{x} + 6x\right) dx = y\ln(x) + 3x^2 + h(y).$$

Taking the y-partial derivative and fitting the remaining term gives

$$\begin{aligned} \frac{\partial F}{\partial y} = \ln(x) + h'(y) &= \ln(x) - 2 \\ h'(y) &= -2 \\ h(y) &= -2y. \end{aligned}$$

So, the general solution is

$$y\ln(x) + 3x^2 - 2y = C.$$

Notice that this can easily be solved for y to give

$$y(x) = \frac{3x^2 + C}{2 - \ln(x)}.$$

2) Suppose that a can of soda is taken out of a refrigerator at 35°F and placed in a room held at 70°F. Then the average temperature of the soda, $T(t)$, satisfies

$$\begin{aligned} \frac{dT}{dt} &= -k(T - 70), \\ T(0) &= 35. \end{aligned}$$

If the temperature of the soda has risen to 45°F after 20 minutes, what is the coefficient k?

The solution to this initial value problem is $T(t) = 70 - 35e^{-kt}$. Using the given information $T(20) = 45$ gives

$$\begin{aligned} 70 - 35e^{-20k} &= 45 \\ e^{-20k} &= \frac{5}{7} \\ -20k &= \ln\left(\frac{5}{7}\right) = -\ln\left(\frac{7}{5}\right) \\ k &= \frac{1}{20}\ln\left(\frac{7}{5}\right). \end{aligned}$$

3) Suppose that a large tank is initially filled with 100 liters of pure water. There is a stream of saline solution with concentration 25 g/L flowing into the tank at a rate of 3 L/min, and we are draining the well-mixed solution out of the tank at a constant rate of 4 L/min.

   a) What is the volume of fluid in the tank as a function of t? When will the tank be empty?

   Since the tank is losing 1 liter every minute, the volume of fluid in the tank is given by $V(t) = 100 - t$. Clearly, the tank will be empty at time $t = 100$ minutes.

   b) Setup an initial value problem that models the total amount of salt in the tank, $A(t)$ as a function of time. You do not have to solve the IVP!

   First, if $C(t)$ is the concentration of salt in the tank for $0 \leq t < 100$, we must have

$$C(t) = \frac{A(t)}{100 - t}.$$

   Since there are 75 grams of salt flowing into the tank every minute, and the solution in the tank is being drained at a rate of 4 L/min, we must have

$$\frac{dA}{dt} = 75 - 4\left(\frac{A(t)}{100 - t}\right),$$
$$A(0) = 0.$$

   The initial condition comes from the information that the tank is initially full of pure water. Even though you were not asked, the solution to this first order linear ODE is

$$A(t) = 25(100 - t) - \frac{(100 - t)^4}{40000},$$

   which means the concentration of salt in the tank is given by

$$C(t) = 25 - \frac{(100 - t)^3}{40000}.$$